A Flexible Framework for Constrained Dix Inversion using Alternating Direction Method of Multipliers

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Summary

We present a novel constrained Dix inversion method that leverages proximal operators within the Alternating Direction Method of Multipliers (ADMM) framework. Addressing problems inherent to Dix inversion, our method robustly manages various noise types in picked RMS velocity functions, including spikes and outliers in measurements that can be distributed in both the space and time domains. Moreover, the inverted model can be adaptively guided towards either blocky or smooth interval velocity fields, making it well-suited for large-scale three-dimensional projects. The effectiveness of the proposed method is demonstrated through comprehensive validation on both synthetic and real data examples.

Introduction

The Dix formula (Dix, 1955) takes root-mean-square (RMS) velocity profiles generally estimated using coherency functionals and converts them into interval or layer velocities. In addition to the inherent shortcomings of its geometrical assumptions, the formula's differencing operator can amplify measurement noise. Moreover, the presence of velocity inversions, may yield a negative radicand, often leading to unrealistic results. The conventional approach to address this limitation involves overly filtering the input data often leads to loss of resolution.

Harlan (1999) proposed a constrained Dix inversion by reformulating the problem as an optimization one. The constrained Dix inversion framework minimizes a cost function that blends project-specific data misfit and regularization terms to describe the statistical properties of the noise in the data and the prior knowledge of the expected interval velocity trends. Such priors frequently necessitate the use of L_1-type norms in both the objective function and regularization terms. While L_2-norm misfit functions are effective for Gaussian noise, but they are inadequate when dealing with outliers. Semi-automated semblance-based velocity analysis, or picking, is notoriously prone to outliers due to several factors including, but not limited to, residual coherent noise (e.g. multiples) that is not entirely suppressed during preprocessing, misidentified layers, or human errors in manual RMS velocity pick extraction.

We choose the Isotropic Total Variation (TV) regularizer due to its directional neutrality, making it particularly well-suited to the complex geometries of real-world geology by preserving blocky layer structure while suppressing noise. Furthermore, applying bound constraints to internal velocity values is essential for achieving geologically meaningful solutions. These requirements lead to formulation of a semi-smooth objective function that integrates both data misfit and regularization terms while enforcing bound constraints. To tackle this challenge, we utilize proximal operators within the Alternating Direction Method of Multipliers (ADMM) framework. A key strength of this approach is its flexibility, as the solver is general and not limited to the constrained Dix inversion framework. Instead, it can accommodate various misfit functions to match several noise statistics, integrate different regularization techniques, and remain agnostic to the specific nature of the linear (or nonlinear) forward operator. This versatility enables the solver to represent prior information more effectively, leading to a more robust and realistic inverted velocity model.

To demonstrate the algorithm's effectiveness under different scenarios, we present both a synthetic and a field data example.

Theory

The root-mean-square (RMS) velocity V_j in discrete form can be written as:

$$V_j^2 = \frac{1}{j+1} \sum_{k=0}^{j} v_k^2, \quad (1)$$

where v_k is interval velocity. Dix formula solves for interval velocities from RMS velocity (Dix, 1955).

$$v_k = \left[kV_k^2 - (k-1)V_{k-1}^2\right]^{\frac{1}{2}} \quad (2)$$

Dix formula involves applying a scaled discrete derivative operator to the data, which inherently acts as a high-frequency amplifier, intensifying noise. This characteristic limits its effectiveness. To address these limitations, various forms of constrained Dix equations have been proposed (Harlan, 1996; Almomin, 2010; Golami and Naeini, 2019). In our approach, we define the unknown variable as $m_i \coloneqq v_i^2$. We then choose a scaled form of the residual. Following (Almomin, 2010; Golami and Naeini, 2019) we rework equation 1 into a scaled residual function, namely:

$$r_j \coloneqq \sum_{k=0}^{j} v_k^2 - (j+1)V_j^2 = \sum_{k=0}^{j} m_k - d_j \quad (3)$$

Using these definitions of residuals and inversion variables, we formulate the constrained Dix inversion problem as follows:

$$\min_{\mathbf{m}} \beta_1 |\mathbf{Mm} - \mathbf{d}|_p + \beta_2 \mathrm{TV}(\mathbf{m}) \quad (4)$$
$$\text{Subject to } \mathbf{m}_l \leq \mathbf{m} \leq \mathbf{m}_u$$

where $\mathbf{M}$ represents the discrete integrator operator for simple summation, $\mathbf{m}_l$ and $\mathbf{m}_u$ are the lower and upper bounds for the unknown variable, β_1 and β_2 are regularization constants. The choice of norm, p, depends on the type of noise in the RMS measurements. The $p = 2$ norm is the natural choice in the presence of additive Gaussian noise. However, the robust $p = 1$ norm is the obvious choice when the additive noise includes outliers. In what follows, we present the formulation for the $p = 1$ case, noting that extending to other norms is straightforward.

For ease of presentation, we follow a 2D formulation but the extension to 3D is trivial. The data and unknown vectors are $\mathbf{d} \in R^{N_z \times N_x}$ and $\mathbf{m} \in R^{N_z \times N_x}$.
The isotropic TV in 2D can be written using matrix norms (Ravasi *et.al*. 2024) as follows:

$$\mathrm{TV}(\mathbf{m}) = |\nabla \mathbf{m}|_{2,1} = \sum_i \sqrt{(D_x \mathbf{m})_i^2 + \left(D_y \mathbf{m}\right)_i^2}$$
(5)

To solve this problem with the ADMM framework, we introduce the slack variables $\mathbf{z_1} \in R^{N_z \times N_x}$, $\mathbf{z_2} \in R^{2 \times N_z \times N_x}$ and $\mathbf{z_3} \in R^{N_z \times N_x}$ and we reformulate the constrained Dix problem in equation 4 as

$$\min_{\mathbf{u}} \beta_1 |\mathbf{z}_1|_1 + \beta_2 |\mathbf{z}_2|_{2,1} + \mathrm{g}(\mathbf{z}_3)$$
$$\text{Subject to } \mathbf{Mm} - \mathbf{d} = \mathbf{z_1}$$
$$\mathbf{Dm} = \mathbf{z}_2$$
$$\mathbf{m} = \mathbf{z}_3 \quad (6)$$

Here, $\mathbf{D} \coloneqq \begin{pmatrix} \mathbf{D}_x \\ \mathbf{D}_y \end{pmatrix}$ where $\mathbf{D}_x$ and $\mathbf{D}_y$ are discrete derivative operators in the *x* and *y* directions, respectively. The function $\mathrm{g}(\mathbf{z}_3)$ represents indicator function for inequality constraints:

$$\mathrm{g}(\mathbf{z}_3) = \begin{cases} 0, & \text{if } \mathbf{m}_l \leq \mathbf{z}_3 \leq \mathbf{m}_u, \\ +\infty, & \text{otherwise.} \end{cases}$$

We follow a scaled augmented Lagrangian approach and introduce the Lagrange multipliers $\boldsymbol{\lambda_1} \in R^{N_z \times N_x}$, $\boldsymbol{\lambda}_2 \in R^{2 \times N_z \times N_x}$, $\boldsymbol{\lambda}_3 \in R^{N_z \times N_x}$ and the augmented Lagrangian constants ρ_1, ρ_2 and ρ_2).The resulting ADMM algorithm recursion at iteration *k* becomes (Boyd *et al*., 2011)

$$\mathbf{m}^{k+1} = \min_{\mathbf{m}} \left\{ \frac{\rho_1}{2} \left| \mathbf{Mm} - \mathbf{d} - \mathbf{z}_1^k + \boldsymbol{\lambda}_1^k \right|_2^2 + \frac{\rho_2}{2} \left| \mathbf{Dm} - \mathbf{z}_2^k + \boldsymbol{\lambda}_2^k \right|_2^2 + \frac{\rho_3}{2} \left| \mathbf{m} - \mathbf{z_3}^k + \boldsymbol{\lambda}_3^k \right|_2^2 \right\}$$

$$\mathbf{z}_1^{k+1} = \min_{\mathbf{z}_1} \left\{ \beta_1 |\mathbf{z}_1|_1 + \frac{\rho_1}{2} \left| \mathbf{Mm}^{k+1} - \mathbf{d} - \mathbf{z}_1 + \boldsymbol{\lambda}_1^k \right|_2^2 \right\}$$

$$\mathbf{z}_2^{k+1} = \min_{\mathbf{z}_2} \Big\{ \beta_2 |\mathbf{z}_2|_{2,1} + \frac{\rho_2}{2} \left|\mathbf{Dm}^{k+1} - \mathbf{z}_2 + \boldsymbol{\lambda}_2^k\right|_2^2 \Big\}$$
$$\mathbf{z}_3^{k+1} = \min_{\mathbf{z}_1} \Big\{ \mathrm{g}(\mathbf{z}_3) + \frac{\rho_3}{2} \left|\mathbf{m}^{k+1} - \mathbf{z}_3 + \boldsymbol{\lambda}_3^k\right|_2^2 \Big\}$$
$$\lambda_1^{k+1} = \lambda_1^k + \rho_1\left(\mathbf{Mm}^{k+1} - \mathbf{d} - \mathbf{z}_1^{k+1}\right)$$
$$\lambda_2^{k+1} = \lambda_2^k + \rho_2\left(\mathbf{Dm}^{k+1} - \mathbf{z}_2^{k+1}\right)$$
$$\lambda_3^{k+1} = \lambda_3^k + \rho_3\left(\mathbf{m}^{k+1} - \mathbf{z}_3^{k+1}\right) \qquad (7)$$

The model update minimizes a quadratic function, while the slack variables $\mathbf{z}_i$ correspond to a proximal or a projection operator with well-known closed-form solutions (Boyd e*t al*., 2011).

Results

We perform synthetic inversion using 2D Marmousi RMS velocity data derived from the true interval velocity model, coarsely sampled at layer boundaries. Gaussian or Cauchy noise is introduced to the data to evaluate the inversion performance under different noise conditions.

We conduct multiple inversion experiments to demonstrate the robustness of the inversion process across various noise characteristics and regularization techniques. For data corrupted with Gaussian noise, the inversion is performed using the plain-vanilla Dix equation 2, and the constrained Dix inversion in equation (6) with two approaches: an L_2-norm misfit combined with first-order Tikhonov regularization and an L_2-norm misfit with total variation (TV) regularization (Figure 1). For data affected by Cauchy noise, we employ inversion with the Dix formula and constrained Dix inversion with two configurations: an L_2-norm misfit with TV regularization and an L_1-norm misfit with TV regularization (Figure 2).

The synthetic Marmousi inversion results reveal several key findings. The plain-vanilla Dix formula proves ineffective at suppressing either Gaussian or Cauchy noise. In contrast, isotropic Total Variation (TV) regularization efficiently reconstructs geological structures, underscoring its value in noise-laden inversion. The L_2-norm misfit function performs well in suppressing Gaussian noise but falters in the presence of outliers. On the other hand, the L_1-norm misfit function adeptly manages Cauchy noise, which mimics outliers, thus demonstrating its appropriateness for datasets with considerable noise variability.

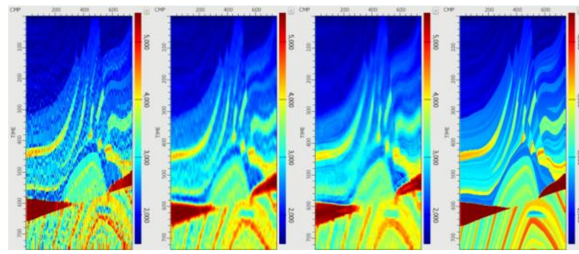

Figure 1 – Inversion results for the synthetic Marmousi test with added Gaussian noise. Interval velocity models from left to right: (a) inversion using the Dix equation 2, (b) constrained Dix inversion in equation 6 using L_2-misfit and first-order Tikhonov regularization, (c) constrained Dix inversion using L_2-misfit and total variation regularization, and (d) the true model. The CMP spacing is 12.5m, and the time is in ms.

A real data inversion example is shown in Figure 3. In this example, the 3D sparse RMS measurements are inverted using the Dix equation 2 and constrained Dix inversion in equation 6 using L_2-misfit and L_1-misfit objective functions with total variation regularization.

The differences in results among the three approaches are less pronounced than in the synthetic example, primarily due to the careful preconditioning of RMS picks to ensure stability in conventional Dix inversion. However, the constrained inversion results exhibit consistency with the baseline inversion, offering minor improvements in resolution and lateral continuity. Notably, as demonstrated in the synthetic example, this approach eliminates the need to precondition the picks.

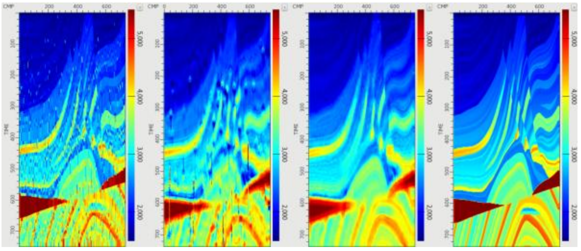

Figure 2 – Inversion results for the synthetic Marmousi test with added Cauchy noise. Interval velocity models from left to right: (a)inversion using the Dix formula, (b) constrained Dix inversion using L_2-misfit and total variation regularization, (c) constrained Dix inversion using L_1-misfit and total variation regularization, and (d) the true model. The CMP spacing is 12.5m and the time is in ms.

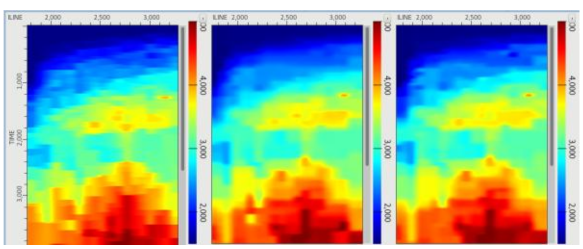

Figure 3 – Inverted velocity estimate using field data, from left to right: (a) inversion using the Dix formula, (b) constrained Dix inversion using L_2-misfit and total variation regularization and (c) constrained Dix inversion using L_1-misfit and total variation regularization. The ILINE and XLINE spacings are 80 and 320m, respectively; time is in ms.

Conclusions

Despite relying on simplified assumptions, Dix inversion remains a widely used tool in the seismic velocity analysis toolbox.

Previous works proposed addressing the limitations of inherent differentiation in the Dix equation by reformulating it as a constrained inverse problem, relying on semi-smooth objective functions, regularizers, and hard inequality constraints.

In this paper, we propose an innovative approach using the Alternating Direction Method of Multipliers (ADMM) in combination with proximal operators to invert for interval velocities. Unlike other specialized Dix inversion methods, our approach is adaptable to a wide range of geophysical challenges. This method can be fine-tuned to meet the unique conditions of a given project.

In the data domain, our approach leverages the robustness of the L_1-norm misfit, making it particularly effective in handling outliers in the picked RMS functions, both temporally and spatially. This contrasts with other published methods, which tend to focus on managing Gaussian-distributed noise in an industry-standard manner by filtering the velocity picks. Furthermore, the ADMM-based method allows for the incorporation of various model regularization techniques to enhance the inverted interval velocities, such as promoting blockiness or incorporating structural guidance.

In conclusion, we present a highly effective solution to the well-known Dix inversion problem. Our method offers versatility and efficiency making it a valuable tool for geophysical velocity analysis.

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