Orientation-invariant wave equations for elastic media with TTI or TORT symmetries

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SUMMARY

Seismic depth-imaging and inversion in complex geology often use wave-equation operators for TTI or TORT symmetries, but these still struggle with numerical instabilities due to incorrect adjoint operators and rapid tilt variations of symmetry axes of elastic tensors. We propose a novel approach using frame bundles to describe heterogeneous anisotropic velocity models, aligning local coordinate frames with the elastic stiffness tensor's tilt. This creates a curved manifold, allowing us to derive wave-equations invariant to symmetry orientation changes of elastic tensors describing the velocity model. Our method provides true adjoint operators, supports various parameterizations of symmetry axes orientations, and interprets tilt variations as additional forces related to the manifold's curvature.

INTRODUCTION

Wave-equation operators designed for elastic media with tilted-transverse isotropy (TTI), or tilted-orthorhombic (TORT) symmetries are nowadays the tool of choice for seismic imaging and inversion in complex geological settings. After their introduction in the early 2000s (Alkhalifah, 2000), these operators gained widespread acceptance in the 2010's following their rapid development by research groups in both oil and service companies (e.g., Fletcher et al., 2009; Zhang and Zhang, 2011; Duveneck and Bakker, 2011; Bube et al., 2012) which have made these operators integral to Reverse-Time Migration (RTM) and Full-Waveform Inversion (FWI) algorithms.

A common feature of these works is the use of directional derivatives to account for the spatial variation of TTI and TORT symmetry axes of the elastic tensors comprising the seismic velocity/earth models. Despite their application and success, directional derivatives do not in general conform to the geometry of the problem, because they are not truly covariant differential operators and thus cannot provide true adjoint operators. Consequently, numerical instabilities arise, particularly when dealing with pronounced spatial variations in the tilt of symmetry axes (Fletcher et al., 2009; Zhang et al., 2009, 2011; Bube et al., 2012; Duveneck and Bakker, 2011). Various ad-hoc solutions, such as smoothing the tilt fields before simulations, have been proposed, but these approaches do not address the root cause of the problem nor are always satisfactory, because they smear important geological features (e.g., bounding faults or salt bodies) which ideally one would like to have as sharp as possible. Thus, the challenge of finding stable adjoint operators for TTI and TORT media is still an open question (e.g., Bube et al., 2016; Louboutin et al., 2018; Le et al., 2019). Addressing the limitations of directional derivatives is thus crucial to meet the growing demand for better and more efficient simulations of seismic waves in ever more challenging subsurface structures.

We argue that the limitations and numerical instabilities of current wave-equations operators for TTI and TORT media stem from overlooking a subtle but fundamental feature of wave propagation in anisotropic media: introducing symmetry axes orientations as model parameters changes the problem's geometry, effectively creating a new curved manifold. Using directional derivatives, in that geometrical sense, implies that one ignores the effects of curvature of the new augmented model space, approximating it as a flat manifold. Such an approximation may work well where the symmetry axes orientations do not significantly vary in space, but it is not general enough to handle situations in which they do.

Our goal is to present a method that generalizes the directional derivatives approach, allowing us to derive symmetry-orientation-invariant seismic wave equations suitable for various numerical solvers, including finite-difference and spectral-element methods (e.g. Afanasiev et al., 2018). These equations have differential operators that are inherently compatible with the geometry of the manifolds representing heterogeneous anisotropic velocity models with TTI or TORT symmetries. This compatibility ensures true adjoint operators even in the presence of rapid spatial variations of symmetry axes. We anticipate that these orientation-invariant wave equations will enhance computational efficiency and provide deeper insights into the physics of elastic wave propagation in heterogeneous anisotropic TTI and TORT velocity models.

Below, we review how tools from differential geometry help represent heterogeneous and anisotropic seismic velocity models as frame bundles that integrate elastic stiffness tensor orientations into our model space geometry. This new concept allows us to write covariant differential operators that account for variable tilts of the elastic tensor's symmetry elements. In the results section, we demonstrate this by rewriting a wave equation used in depth imaging as a fully covariant wave equation, invariant to spatial changes in the orientation of the elastic tensor.

Orientation-invariant wave equations for TTI and TORT

THEORY

Our development draws on fundamental concepts from differential geometry that also underlie other physical theories such as General Relativity and Gauge Theory: manifolds, fiber bundles, and connections. We conceptualize anisotropic velocity models as Riemannian manifolds described by six coordinates: three spatial coordinates and three for the local orientation of the elastic tensor. This parameterization is locally trivial but globally complex, akin to how the Earth's surface appears locally flat (i.e., Euclidean) but is globally curved. Thus, anisotropic velocity models are not six-dimensional Euclidean spaces and require a more sophisticated construction. Fiber bundles provide this framework by generalizing Cartesian products, allowing the creation of complex manifolds by combining simpler ones (e.g., Tu, 2017). In our case, this combination—called a fiber bundle—comes from attaching a copy of the orientation grid (a manifold on its own) to every point in our spatial coordinate grid (the base manifold). These fibers can "twist" as they attach to the base space, creating intricate global structures, which are stitched together smoothly by transition functions. The bundle is characterized by the combined total space, the base space, and a projection map that matches points from the total to the base space. Classic illustrations of fiber bundles are the cylinder and Moebius strip. Both can be represented by the combination of a circle as a base space with the real line representing the fibers attached to each point of that circle. Locally, both the cylinder and Moebius strip look like Euclidean spaces, but in the Moebius strip, the orientation of the fibers changes along the circle, resulting in a total space that is not a cylinder (see Figure 1). Finally, connections extend the notion of directional derivatives to fiber bundles. In curved spaces, connections allow the parallel transport of vectors and tensors along curves in the base space, enabling their comparison and differentiation across different fibers.

In our description of TTI or TORT velocity models, we use the so-called frame bundle, where the fiber represents the space of all possible orientations of the frame describing the local coordinate axes (Tu, 2017). Our total space model is characterized by a local frame at each point in the base space, taken to be the same frame in which elastic tensors are given by their smallest number of parameters (five in the case of TTI and nine for TORT; see Figure 2). This abstraction enables us to deploy the powerful tools of fiber bundles to understand elastic wave propagation in anisotropic models in a new and useful way. It allows us to define a (gauge) connection operator that is (1) compatible with the metric of our 6D manifold (ensuring true adjoint operators in wave propagation) and (2) automatically accounts for spatial changes in orientation of TTI or TORT symmetry axes when taking (covariant) derivatives, making it unnecessary the rotation of elastic tensors during wave propagation.

The fiber bundle approach splits the tangent space of the total space into two complementary pieces, one along fibers and another perpendicular to them. Consequently, the gauge connection operator acting on a vector or tensor field T defined over the total space manifold and taken along a vector is given by the summation of two operators

$$D_X T = \nabla_X T + \sum \omega(X) T \,, \tag{1}$$

where ∇ represents the part of the connection accounting for changes in position in the base space, and ω for changes in the orientation of the elastic tensor (i.e., in the local frame orientations). The connection 1-form, ω, acts over each index of T, thus the summation symbol in the definition of $D_X T$. Incidentally, (differential) 1-forms are dual to vectors, i.e., on manifolds they represent linear functions acting on vectors to produce scalars. For simplicity and to highlight the contribution of spatial variation of orientations of anisotropic axes into wave propagation, we take the base space to represent a Euclidean space, such that $\nabla_X = \partial_X$, i.e., changes along base space directions are given by directional derivatives. Technical details aside, the operator ω in the frame bundle has a very specific form: it is a skew-symmetric 3×3 matrix, whose elements are given by $\omega = h^{-1} dh$, where h is the rotation/attitude matrix determining the orientation of the elastic tensors at each point in the anisotropic velocity model. Thus, ω is a model parameter, independent of propagation, which may be precomputed or obtained on the fly, once the attitude matrix is set, regardless of its parameterization using Euler angles, quaternions, or other methods.

RESULTS

We can now recast the wave equations for media with TTI or TORT symmetries using the frame-bundle approach. In practice, this involves replacing any partial or directional derivatives with the gauge connection D and ensuring the resulting equation is frame independent. For example, starting from a wave equation similar to those found in Fletcher et al. (2009) and Duveneck and Bakker (2011) written in terms of stress tensor σ components and considering constant density, we go from

$$\partial_t^2 \sigma_{ij} = A_{ij}{}^{kl} \partial_l \partial^r \sigma_{kr} \tag{2}$$

to

$$\begin{aligned} \partial_t^2 \sigma_{ij} &= A_{ij}{}^{kl} D_l D^r \sigma_{kr} = A_{ij}{}^{kl} D \left(\mathrm{tr}\left(\mathrm{D}\sigma\right)\right)_{kl} \\ \partial_t^2 \sigma_{ij} &= A_{ij}{}^{kl} \left(\mathrm{tr}\left(\mathrm{D}\left(\mathrm{D}\sigma\right)\right)\right)_{kl} \end{aligned} \tag{3}$$

where A is the density-normalized elastic stiffness tensor and summation over an index is denoted by its appearance in one upstairs and one downstairs position. After the second equality sign in equation 3, we switched to a more abstract notation to show that the sequence of covariant derivatives represents the gradient of the divergence of the stress tensor. Because D is compatible with the metric (i.e., $Dg = 0$, where g is the metric tensor of the manifold), it commutes with the trace operator and so we reach the expression with the term $D(D\sigma)$. This is not a tensorial quantity, thus not frame-independent. It is, nevertheless, part of the Hessian of the stress tensor,

$$D^2_{X,Y}\sigma = D_X(D_Y\sigma) - D_{D_XY}\sigma\,, \tag{4}$$

which is tensorial thanks to the extra term $D_{D_XY}\sigma$ (with X and Y being vectors). Because equation 2, our starting point, is set up in a Euclidean manifold—where the second term of the Hessian is identically zero—our translation from directional derivatives to covariant ones requires that $\partial^l\partial_s\sigma_{kr}$ be understood as the full Hessian. Thus, equation 2 should be written as the full Hessian. Thus, equation 3 should be written as

$$\partial_t^2\sigma_{ij} = A_{ij}{}^{kl}\left[\mathrm{tr}(\mathrm{D}^2\sigma)\right]_{kl} = -A_{ij}{}^{kl}\Delta\sigma_{kl}\,, \tag{5}$$

where Δ is the Laplacian operator compatible with the metric of the manifold, which is not simply the summation of second partial derivatives. In curved spaces, like our frame bundle, second covariant derivatives do not commute, but their difference defines a crucial quantity that measures the curvature of the total space itself, the curvature tensor $R(X,Y)$, i.e.,

$$D^2_{X,Y}\sigma - D^2_{Y,X}\sigma = \sum R(X,Y)\sigma\,, \tag{6}$$

where the summation is over the indices of the stress tensor, analogously to the behavior of the connection 1-form ω in the definition of covariant derivative. Equation 6 underscores that the Laplacian in equation 5 contains terms related to the curvature tensor, since one could write

$$\Delta\sigma = -\,tr\left[D^2_{X,Y}\sigma\right] = -tr\left[D^2_{Y,X}\sigma + \sum R(X,Y)\,\sigma\right]. \tag{7}$$

This is a physically insightful result. First, it reinforces the limitation of the directional derivative approach because terms with the connection 1-form ω (found in the covariant derivative D) and the curvature R are ignored. Second, it links differences in orientation of symmetry axes in TTI and TORT models to extra accelerations (akin to tidal forces) in the wave equation, coming from the action of the curvature (tensor) on the stress perturbations describing the waves. Lichnerowicz (1961) provides the following expression for the Laplacian of second-order tensors in Riemannian manifolds

$$(\Delta\sigma)_{kl} = -D^iD_i\sigma_{kl} + Ric_{ik}\sigma^i_l + Ric_{il}\sigma^i_k - 2R_{ijkl}\sigma^{ij}\,, \tag{8}$$

where Ric is the Ricci tensor, obtained from contraction of two indices of the curvature tensor R. The Lichnerowicz Laplacian in equation 8 is self-adjoint and it neatly separates the cuvature effects when computing Laplacians. The term $D^iD_i\sigma_{kl}$, the *rough Laplacian*, is what one would get if the manifold were flat. It comprises the familiar $\partial^i\partial_i\sigma_{kl}$ terms plus terms with the connection 1-form ω. Note that the Laplacian returns a tensor of the same dimension and with the same symmetries of the input tensor. In our frame bundle, this means that $\Delta\sigma$ can also be represented by a 3×3 symmetric matrix. Both the curvature and the Ricci tensors are evaluated to where the wavefield is, making finite-difference stencils redundant for their numerical evaluation and thus having negligible impact on numerical computations during propagation. The curvature tensor R can be calculated from the metric tensor of the bundle but is more efficiently computed from the connection 1-form. Indeed, the covariant derivative of ω results in another skew-symmetric 3×3 matrix Ω,

$$\begin{aligned}\Omega(X,Y) &= (D\omega)(X,Y)\,,\\ \Omega(X,Y) &= \partial_X\omega(Y) - \partial_Y\omega(X) + \omega(X)\wedge\omega(Y)\,,\end{aligned} \tag{9}$$

where the wedge product $\wedge$ is the multiplication operation for differential forms. In short, the wedge product is the tensor product followed by antisymmetrization based on the permutation of indices of each term. In equation 9 the wedge product between the matrices of 1-forms then is a shorthand for the antisymmetrization of the implied matrix multiplication:

$$\omega(X)\wedge\omega(Y) = \omega(X)\omega(Y) - \omega(Y)\omega(X) \tag{10}$$

A standard result of differential geometry then connects Ω to the curvature tensor R, because each element of this matrix is in fact a function of the components of the curvature tensor in a local coordinate frame x^i (e.g., Petersen, 2006):

$$\Omega^i{}_j = \frac{1}{2}R^i_{jkl}dx^k\wedge dx^l\,. \tag{11}$$

Given this direct link to the curvature tensor, Ω is called the 2-form curvature (of the connection). Like ω, it can be computed once a velocity model is known, and a parameterization of the fibers is chosen.

In equation 11, the elements of the matrix Ω are sums of 2-forms with coefficients from the curvature tensor R expressed in terms of the 2-form bases $dx^k\wedge dx^l$. These bases are unit areas defined by wedging the basis 1-forms dx^k and dx^l related to coordinates x^i $(i = 1,\ldots,N)$ used to parameterize the frame bundle. This illustrates that the wedge product combines lower-dimensional forms into higher-dimensional ones ensuring that the resulting differential k-forms are also alternating multi-linear functions of k input vectors. This encodes oriented lines, planes, volumes, and higher-dimensional counterparts in k-forms streamlining calculus and therefore wave propagation in general manifolds. Thus, $dx^k \wedge dx^l$ is a 2-form that will take two vectors and output the oriented projection of the area spanned by them onto the plane of the 2-form, with the sign of the result changing whenever one swaps the order of the input vectors.

With all the elements above, one is now able to write covariant wave equations for TTI and TORT media as we did for example equation 2. The key point is that the frame bundle provides the means to write wave equations that are invariant to changes in the orientation of the symmetry axes of the elastic tensors.

CONCLUSIONS

We have introduced a new, mathematically rigorous approach to wave propagation in heterogeneous elastic and anisotropic models with TTI or TORT symmetry. By conceptualizing velocity models as frame bundles, where local coordinate frames align with the natural frame of elastic tensors, we generalize and improve upon the current practice of using directional derivatives. This approach encodes frame rotations directly into covariant wave equations, ensuring differential operators are compatible with the metric of the new manifold formed by combining spatial dimensions with the orientation manifold. This method should enhance stability and reduce numerical artifacts in wavefield simulations, potentially improving reflectivity and velocity model estimation in inverse problems like LSRTM or FWI. The frame bundle approach also provides several insights and benefits for wave propagation in anisotropic media: it separates the effects of the underlying space geometry from the elastic tensor orientation, lends physical meaning to orientation changes of symmetry elements as new tidal forces, introduces a new model parameter constraint in the form of a curvature tensor for inversion problems, and opens avenues for developing more accurate and efficient solvers that can adaptively account for frame rotations. Additionally, it encourages further investigation into orientation parameterization, such as using Euler angles or quaternions.

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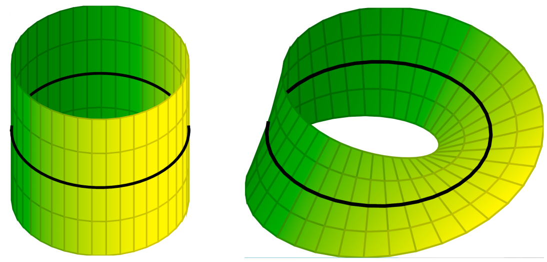

Figure 1: A cylinder and a Moebius strip are two fiber bundles created by attaching copies of the real line (the fibers) to a circle (the base space). Locally they can be represented by the same flat space, but their global topologies are widely different because how the fibers twist in the Moebius strip.

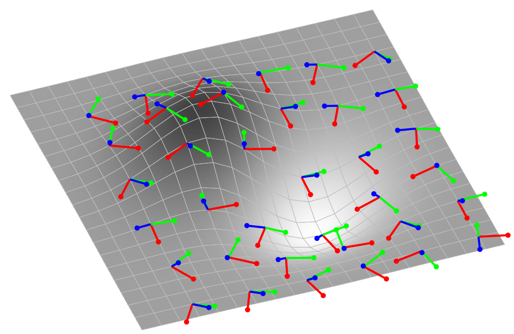

Figure 2: Frame bundle concept for TTI and TORT models. The gray 2-D surface represents the computational grid, which determines the spatial discretization of the model using general curvilinear grids. The red, green and blue axes show local frame orientations that are attached to the base space but align with the elastic tensor symmetry elements at each grid position, not to the geometry of the underlying base space. This way elastic tensor orientation is decoupled from the underlying spatial geometry.